DID PEIRCE HAVE HILBERT'S NINTH AND TENTH PROBLEMS?

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Abstract. In the manuscript "Logical Studies of the Theory of Numbers" (*ca*. 1890), Peirce asks whether there is an algorithm for finding solutions to equations in number theory. He thus anticipates, in a more general way, David Hilbert's Tenth Problem, posed at the International Congress of Mathematicians in 1900, of determining whether there is an algorithm for solutions to Diophantine equations. Peirce proposes translating these equations into Boolean algebra, but does not show how to use that to solve equations. In 1931 Gödel proved that number theory is incomplete, and in 1970 Yurii Matiyasevich gave a negative answer to Hilbert.

Sommaire. Dans le manuscrit «Logical Studies of the Theory of Numbers» , Peirce demande s'il existe un algorithme pour trouver des solutions aux équations en théorie des nombres. Il a donc anticipe, d'une manière plus générale, dixième problème de David Hilbert, posée au Congrès International de mathématiques en 1900, de déterminer s'il existe un algorithme pour trouver des solutions aux équations diophantienne. Peirce propose de traduire ces équations en algèbre booléenne, mais n'affiche pas comment l'utiliser pour résoudre des équations. En 1931 Gödel s'est avéré que la théorie des nombres est incomplète, et en 1970, Yurii Matiyasevich a donné une réponse négative de Hilbert.

Zusammenfassung. In das Manuskript "Logical Studies of the Theory of Numbers" fragt Peirce, ob es sich um ein Algorithmus für die Suche nach Lösungen für Gleichungen in Zahlentheorie. Er rechnet damit, auf eine allgemeinere Weise David Hilbert zehnten Problem, bei der Internationalen Kongress der Mathematiker im Jahr 1900, der bei der Beurteilung, ob es ein Algorithmus für Lösungen für diophantische Gleichungen gestellt. Peirce schlägt vor, diese Gleichungen in Boole'schen Algebra übersetzen, aber wie, die zum Lösen von Gleichungen verwendet nicht angezeigt. 1931 Bewiesen Gödel, dass Zahlentheorie unvollständig ist, und 1970 Yurii Matiyasevich eine negative Antwort auf Hilbert gab.

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In the manuscript "Logical Studies of the Theory of Numbers" (*ca*. 1890) [Peirce *ca. 1890*], Charles Sanders Peirce (1839–1914) asks whether there is an algorithm for finding solutions to equations in number theory. He also asks whether there is an algorithm for determining if there are proofs in number theory. "The object of the present investigation," he writes [Peirce *2010*, p. 55], "is to analyze carefully the logic of the theory of numbers. I especially desire to clear up the question of whether there can be fundamentally different ways of proving a theorem from given premises; and the law of reciprocity seems to be instructive in this respect. I also wish to know whether there is not a regular method of proof in higher arithmetic, so that we can see in advance precisely how a given proposition is to be demonstrated." He thus seems to anticipate, in a more general way, David Hilbert's Tenth Problem, posed at the International Congress of Mathematicians in 1900, of determining whether there is an algorithm for solutions to Diophantine equations. Peirce proposes translating these equations into Boolean algebra, but does not show how to use that to solve equations.

Like much of his work, Peirce failed to follow through with this project. The manuscript was found interpolated into an unsent letter to an unknown correspondent.

In this manuscript, Peirce (1839–1914) wants to know if there exists a procedure for solving equations in number theory. He does not specify in particular the solution of Diophantine equations, that is of polynomial equations of any number of unknowns, whose solutions are to be determined (or, for example, finding Mersenne numbers) in particular. He appears to be working with generalized polynomial equations of no particular degree. For example, a linear Diophantine equation (in two variables) is an equation of the general form $ax + by = c$, where solutions with a, b, and c integers are to be determined. (For a history of Diophantine equations, see [Bashmakova

1997]. This should be supplemented by [Davis *1973*] and [Davis & Hersh *1973*].) The more specific, of in some way general, question being asked is whether there is a procedure for proving a theorem (equation) from its premises (axioms of number theory). The multiplicity of proofs of the Law of Reciprocity, and its increased generalizations, suggests by means of example a clue of finding whether there are procedures for solving equations and, if so, of how, if there is more than one procedure, of selecting from among them.

The first formulation of his questions asks whether there may be different ways of proving equations from given premises, and he suggests that the Law of Reciprocity might offer a hint regarding the answer to this question. The Law of Reciprocity (also known as the Law of Quadratic Reciprocity) was stated by Leonhard Euler (1707–1783), although he was unable to prove it. (The statement and a sketch of Euler's informal argument is given in a letter to Christian Goldbach (1690–1764) of 28 August 1742 [Euler *1742*]; the letter appears in [Fuss *1843*, I, 143–153] and [Jušk̆evič & Winter *1965*]; an English translation of the letter and a formal proof in modern terminology and notation is given in [Edwards *1983*].) Also called the *aureum theorema* (golden theorem) by Carl (Karl Friedrich) Gauss (1777–1855), it states that: If p and q are distinct odd primes, then the congruences

$$x^2 \equiv q \pmod{p}$$
$$x^2 \equiv p \pmod{q}$$

are both solvable or both unsolvable unless both p and q leave the remainder 3 when divided by 4 (in which case one of the congruences is solvable and the other is not). Written symbolically,

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}$$

where

$$\frac{p}{q} \equiv \begin{cases} 1, & \text{for } x^2 \equiv p (\text{mod } q) \text{ solvable for } x \\ -1, & \text{for } x^2 \equiv p (\text{mod } q) \text{ not solvable for } x \end{cases}$$

is known as a *Legendre symbol*, from Legendre, who gave the first complete formulation of the Law and expressed it in terms of the Legendre symbol, due to Adrian Marie Legendre (1752–1833).

Euler stated the theorem in 1783 without proof (see [Edwards *1988*]). Legendre first set it forth in "Recherches d'analyse indéterminée" [Legendre *1785*] and was the first to publish a proof, in his *Essai sur la théorie des nombres* [Legrendre *1798*], but it was fallacious. (See [Pieper *1997*] and [Weintraub *2009*] on Legrendre's work on the law.) In 1796, Gauss became the first to publish a correct proof ([Gauss *1801*, §41, 141–145; see esp. p. 144; see [Frei *2007*, 180] for the account, for based on Gauss's diary; Gauss's proofs of his "Fundamental Theorem", the Quadratic Reciprocity Law, were presented in sect. 4 of his *Disquisitiones arithmeticae* [Gauss *1801*]. For details of the history of the Law, see [Lemmermeyer *2000*].) The quadratic reciprocity theorem was Gauss's favorite theorem from number theory, and he devised no fewer than eight different proofs of it over his lifetime. Another proof was added by Ferdinand Gotthold Max Eisenstein (1823–1852) [Eisenstein *1844*]. (For more see [Edwards *1982*, pp. 31–43] and [Laubenbacher & Pengelley *1994*]). By the mid-1970s, there were several hundred proofs of the Law of Quadratic Reciprocity (see, e.g. [Adams & Goldstein *1976*, 12]. Meanwhile, Ernest Eduard Kummer (1810–1893) [Kummer *1847a*, *1847b*] generalized the law by introducing ideal numbers into the field of complex numbers after his efforts to solve Fermat's Last Theorem led him to recognize that attempts to prove the theorem broke down because the unique factorization of integers did not extend to other rings of complex numbers. He thus attempted to restore the uniqueness of factorization by introducing ideal numbers. Not only has his work been most fundamental in work relating to Fermat's Last Theorem, since all later work was based on it for many years, but the concept of an ideal allowed ring theory, and much of abstract algebra, to develop.

The genus theorem states that the Diophantine equation

$$x^2 + y^2 = p$$

can be solved for p a prime if and only if $p \equiv 1 \pmod{4}$ or $p = 2$.

It was during this period that Peirce was writing definitions for the *Century Dictionary*, and among the definitions for which he was responsible was the Law of Reciprocity. In the *Dictionary*, Peirce defines "quadratic reciprocity”, as a subentry of “quadratic” as: “the relation between any two prime numbers expressed by the law of reciprocity” [Peirce *1889-1910c*], and refers the reader to the definition of “law”, with a subentry on the “law of reciprocity of prime numbers” which reads [Peirce *1889-1910a*]: “the proposition that if p and q are two prime numbers, then, if p is a quadratic residue of q, q is also a quadratic residue of p, unless both leave the remainder 3 when divided by 4, when, if p is a quadratic residue of q, then q is not a quadratic residue of p.” Although these articles are unsigned, we are are certain that these definitions were among his contributions.[1]

The subject of the multiplicity of proofs of the Law of Quadratic Reciprocity was, as a matter of fact, the subject of discussion during the period when that issue was raised by Peirce. For example, in his review of the new textbook on number, *Die Elemente der Zahlentheorie* [Bachmann *1892*] by Paul Gustav Heinrich Bachmann (1837–1920) for the *Bulletin of the New York Mathematical Society*, Jacob William Albert Young (1865–1948) wrote [Young *1894*, 218–220]:

> SECTION III. *Quadratic Residues.*—Quadratic residues are defined, Euler’s criterion for the possibility of the congruence $x^n \equiv n$ (mod p) is set up, Legendre’s symbol is introduced, the congruence $x^n \equiv$ (mod m) is considered for composite moduli of various forms, and then the problem is attacked : “To determine of what moduli a given number n is residue (or non-residue)” Restricting consideration first to prime moduli, we ask successively, “Of what odd primes is the numbers, — 1, 2, any other prime q, residue (or non-residue)?” For the first two cases the question is satisfactorily and readily answered, while the third leads to Legendre’s *law of reciprocity* of quadratic residues. The consideration of this important theorem is begun by proving the “Gaussian lemma,” that $\left(\frac{n}{p}\right) = (—1)^{\mu}$, where μ is the number of negative numbers in the series of absolutely least residues (mod p) of
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> $$n,\ 2n,\ 3n,\ \ldots \frac{p—1}{2}.n.$$
>
> Before proceeding to a proof of the law of reciprocity, the author classifies the proofs which have been given into four categories:
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> *First Category.*—The first proof of Gauss, simplified by Dirichlet, constitutes a category by itself. Of all the proofs ever given, it attacks the problem most directly, and by induction achieves the result through simple conclusions from the idea of a quadratic residue.
>
> *Second Category.*—The proofs of this category are based upon the theory of quadratic forms. To this category belong the second proof of Gauss, those of Kummer, and that of Legendre (incomplete).
>
> *Third Category.*—Proofs directly connected with the division of the circle. They are exemplified by the fourth and the sixth proof of Gauss.
>
> *Fourth Category.*—Proofs based on the Gaussian lemma. Examples, the third and the fifth proof of Gauss.
>
> As the first category is treated in full in Dedekind-Dirichlet’s text-book, and as the second and the third category are not elementary, they are (with one exception in the second category to be noticed later) excluded from detailed consideration.
>
> Of the fourth category, the proof of the pastor Zeiler is given.* This proof, though simple and elegant, has not before, as far as the reviewer knows, been presented in a textbook on the theory of nurnbers.† There follow Jacobi’s generalization of Legendre’s symbol and the generalized law of reciprocity, Eisenstein’s algorithm for the determination by $\left(\frac{m}{n}\right)$, m and n being any two relatively prime integers of which at least one is positive, while with Schering'' and Kronecker’s generalization of the Gaussian lemma and, based on it, Schering’s proof of the generalized law of reciprocity, and Kronecker’s representation of the symbol $\left(\frac{Q}{P}\right)$, (Q and P being any two positive, odd, relatively prime integers) as the sign of certainproducts and the consequent simplification of Gauss’ third proof, and as an application of Kronecker’s representation another proof by Kronecker of the law of reciprocity, bring us into the field of the most recent researches in this subject.
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> * *Berl. Montasber.* 1872.
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> † It may be of interest to give in this connection a list of the proofs presented in various text-books:
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> *Tschebyscheff,* Gauss III.

Dedekind-Dirichlet, Gauss I, III, IV.
Serret (*Alg. Sup.*), Gauss III.
Bachmann (Kreistheilung), { Gauss VI, Eisenstein (*Crelle* 27). / Liouville, Eisenstein (*Crelle* 29). / Gauss IV (sketch).
Wertheim, Gauss V, Eisenstein (*Crelle* 28).
Mathews, Gauss I, III, Eisenstein (*Crelle* 27, in part).

Peirce was a member of the mathematical society and there is no reason to think that he did not read these lines. What remains problematic are: (a) whether he may have written about the multiplicity of proofs of the Law before or after reading Young's remarks; and (b) whether, regardless of when, or even if, he read Young's remarks, he was not already aware of the multiplicity of proofs, and, (c), if so, he considered it in some respect to be a difficulty. These issues are problematic to the extent that the manuscript "Logical Studies of Number Theory" [Peirce *ca. 1890*] is dated only approximately, the best current approximation being *circa* 1890, primarily because the characteristics of the paper on which the manuscript was written was used by Peirce largely, if not entirely, in 1890. What we do know from the archival evidence is that the manuscript "Logical Studies of the Theory of Numbers" [Peirce *ca. 1890*] was extracted from an undated letter too an unknown correspondent, that the datation of *ca*. 1890 officially assigned by the Peirce Edition Project is only approximate, but that Peirce apparently was also reading Moritz Benedikt Cantor's (1829–1920) history of mathematics at about that time. In fact, of course the second volume of the first edition of Cantor's *Vorlesungen über Geschichte der Mathematik* first appeared in print in 1892, and it is there [Cantor *1892*, 772–773], and in turn referencing Charles Henry (1859–1926) [Henry *1879*] and Paul Tannery (1843–1904) [Tannery *1883*], that Fermat's Last Theorem is discussed, in connection with Fermat's study of Diophantos of Alexandria (*ca*. 200–*ca*. 284). Thus, we can at most assume, with respect to the chronological reference for Peirce's composition of his manuscript, that it was composed some time very early in the decade of the 1890s, and is more likely, then, to have been inspired by a reading of Cantor's remarks on Fermat's Last Theorem than by the questions posed by David Hilbert (1862–1943) at the International Congress of Mathematicians in Paris in July 1900, which did not appear in print until 1901 and in English until the middle of 1902.

The history of the Law of Reciprocity and of the multiple proofs of it given by Gauss led Peirce to the next formulation of his question: "whether there is not a regular method of proof in the higher arithmetic", i.e. number theory, "so that we can see in advance precisely how a given proposition is to be demonstrated." Thus, Peirce seems to be asking whether there is an algorithm for determining whether (and if so, how) to determine for any given Diophantine equation, whether that equation has a solution in the integers.

Diophantine equations are generalizations of the Pythagorean formula, and have the form

$$a^n + b^n = c^n$$

where *a*, *b*, *c*, and *n* are all integers. Pierre de Fermat (1601–1665), in the margin of his copy of the *Arithmetica* of Diophantos of Alexandria (*ca*. 200–*ca*. 280 A.D.), wrote against this that he had a proof that there are no integers for *n* greater than 2 such that there is a solution in the positive integers to the Diophantine equation. Thus Fermat (at [Diophantus *1621*, 85, problem II.8]: "Cubum autem in duos cubos, aut quadrato-quadratum in duos quadrato-quadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet."[2] This has come down to us as Fermat's Last Theorem. Whereas the result had been proven over the nearly 350 years since Fermat stated his theorem for specific values of *n*, it was not until Andrew Wiles (b. 1953), with the aid of Richard Taylor (b. 1962) for a special recalcitrant case involving Hecke algebras [Taylor & Wiles *1995*], gave his proof that a result, known as the Taniyama-Shimura (and sometimes the Taniyama-Shimura-Weil) Conjecture, which was known to imply Fermat's Last Theorem, has there been a proof of the general theorem [Wiles *1995*]. (For a sketch accessible to mathematicians who are not necessarily algebraic number theorists, see [Faltings *1995*].) The proof involves results in number theory, in particular algebraic number theory, and especially the theory of elliptical functions, that were only developed over the course of those three hundred fifty years since Fermat, many of those only in the twentieth century, and almost all of them by mathematicians working at attempts to prove Fermat's theorem.

The Law of Quadratic Reciprocity, like Fermat's Last Theorem, is Diophantine. Moreover, the concepts involved contribute to our consideration of various Diophantine equations. In particular, we can use the Law of Quadratic Reciprocity to determine for specific Diophantine equations which are and which are not solvable (see

[Adams & Goldstein *1976*, 132–135] for examples and proofs).

In 1900 at the International Congress of Mathematicians in Paris, David Hilbert listed twenty-three unsolved problems, and set the goal for twentieth-century mathematicians of solving these problems. The Ninth Problem directly concerns the Law of Reciprocity [Hilbert *1901*]:

> **9. Beweis des allgemeinsten Reziprozitätsgesetzes im beliebigen Zahlkörper**
> *Für einen beliebigen Zahlkörper soll das Reciprocitätsgesetz der l-ten Potenzreste bewiesen werden,* wenn l eine ungerade Primzahl bedeutet und ferner, wenn l eine Potenz von 2 oder eine Potenz einer ungeraden Primzahl ist. Die Aufstellung des Gesetzes, wie die wesentlichen Hülfsmittel zum Beweise desselben werden sich, wie ich glaube, ergeben, wenn man die von mir entwickelte Theorie des Körpers der l ten Einheitswurzeln {Bericht der Deutschen Mathematiker-Vereinigung über die Theorie der algebraischen Zahlkörper, Bd. IV, 1897. Fünfter Teil} und meine Theorie {Mathematische Annalen, Bd. 51 und Nachrichten der K. Ges. d. Wiss. zu Göttingen 1898} des relativ-quadratischen Körpers in gehöriger Weise verallgemeinert.
>
> **9. Proof of the most general law of reciprocity in any number field**
> *For any field of numbers the law of reciprocity is to be proved for the residues of the l-th power,* when *l* denotes an odd prime, and further when *l* is a power of 2 or a power of an odd prime.
> The law, as well as the means essential to its proof, will, I believe, result by suitably generalizing the theory of the field of the *l*-th roots of unity, [Hilbert *1897*] developed by me, and my theory of relative quadratic fields [Hilbert *1899*].

It is mathematically equivalent to Peirce's expectation that the Law of Quadratic Reciprocity, having multiple proofs, can be generalized as applicable to any pair of primes.

The problem was partially solved by Emil Artin (1889–1962) by establishing the Artin reciprocity law which deals with abelian extensions of algebraic number fields [Artin *1927*]. Additional contributions were made by Igor Rostislavovich Shafarevich (b. 1923), who provided the widest generalization of Reciprocity to date [Shafarevich *1950*].

Hilbert's Tenth Problem asks [Hilbert *1901*]:

> **10. Entscheidung der Lösbarkeit einer diophantischen Gleichung**. Eine diophantische Gleichung mit irgendwelchen Unbekannten und mit ganzen rationalen Zahlkoefficienten sei vorgelegt: *man soll ein Verfahren angeben, nach welchen sich mittels einer endlichen Anzahl von Operationen entscheiden lässt, ob die Gleichung in ganzen rationalen Zahlen lösbar ist.*
>
> **10. Determination of the solvability of a diophantine equation**
> Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: *to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers.*

and is mathematically equivalent, in its core and essence, to Peirce's more general question, of whether there exists an algorithm for finding a solution for every Diophantine.

In 1931 [Gödel *1931*] Kurt Friedrich Gödel (1906–1978) arrived at his incompleteness theorems, and proved (1) that there are sentences of number theory that are true but unprovable within the axiom system in which they occur. He also proved (2) that any such consistent axiomatic system cannot prove its own consistency. This means that there are formally undecidable sentences of number theory. As examples of these Gödel explicitly mentioned three number-theoretic results, namely: Fermat's Last Theorem; the Goldbach Conjecture, proposed by Christian Goldbach (1690–1764) in a letter to Leonhard Euler of June 7, 1742, in which Goldbach states that "at least it seems that every number that is greater than 2 is the sum of three primes [Goldbach *1742*; Dickson *1919-23*, I, 421]; and the Riemann Hypothesis, proposed by Georg Friedrich Bernhard Riemann (1826–1866) [Riemann *1859*], according to which the nontrivial Riemann zeta function zeros, i.e., the values of s other than –2, –4, –6, ... such that $\zeta(s) = 0$ (where $\zeta(s)$ is the Riemann zeta function) all lie on the "critical line" $\sigma = \mathcal{R}[s] = ½$ (where $\mathcal{R}[s]$ denotes the real part of s). A more general statement known as the Generalized Riemann hypothesis conjectures that neither the Riemann zeta function nor any Dirichlet *L*-series has a zero with real part larger than ½. As yet, examples pile up for specific cases, but no definitive proof of either of these two results in their most general formulations has been obtained.

Peirce for his part followed up the first formulation of his question regarding number theory, which was mathematically equivalent to asking for a generaliziation of the Law of Quadratic Reciprocity, and whether there is an algorithm for dealing with equations involving quadratic residues, with a more general formulation, to ask if there is a "regular method of proof in the higher arithmetic, so that we can see in advance precisely how a given proposition is t be demonstrated."

On the presupposition that this is a reference back to Diophantine equations specifically, this is mathematically equivalent to Hilbert's Tenth Problem, asking whether there exists an algorithm for solving Diophantine equations in the positive integers. Although Peirce does not explicitly give an answer to this question, he proposes a method of translating number-theoretic equations into the language of the algebra of relatives. Whether recommendation of this procedure indicates on Peirce's part a belief that the answer to the question of the existence of an algorithm for deciding the solvability of Diophantine equations, is problematic. We clearly cannot read Peirce's mind, in particular since there is no explicit answer in the manuscript beyond an illustration of how to rewrite number-theoretic equations algebraically. Moreover, the manuscript does not offer instructions, or even a hint, of how, once a translation has been completed, the equations would be handled. It would also, therefore sheer speculation to conclude to that Peirce did not undertake, let alone complete, this task undertaken in this manuscript because he anticipated that the answer to the question would, as Yurii Vladimirovich Matiyasevich (b. 1947) [Matiyasevich *1970*] proved nearly eighty years later, turn out to be negative.

Elsewhere, there is material that tends towards a preliminary conjecture that Peirce held that the "logicization" of number theory would provide an algorithm for proving all true sentences of number theory. In the manuscript "Axioms of Number" [Peirce *1880/81*] and the closely related "The Axioms of Number" [Peirce *ca.1881?*],[3] Peirce provided fifteen axioms (or assumptions) of arithmetic which provide a definition of "positive, discrete number" and from which, Peirce thought, every proposition of the theory of numbers may be deduced by formal logic, along with definitions of "addition" and "multiplication." In related manuscripts, in particular the "Numerical Equations" [Peirce *n.d.*], Peirce is concerned with the method of getting all the roots when their moduli are all different, and in "Analysis of some Demonstrations concerning definite Positive Integers" [Peirce *1905-6*], in which Peirce offers his existential graphs as a means of analyzing arguments of logic, and on the versos of some pages are contained notes for definitions for the *Century Dictionary*, there is a hint that Peirce was considering logic as a means of providing an algorithm for the solution of Diophantine equations.

The "Logical Studies of Number Theory" manuscript can be seen (as [Houser *2010*, xliii, n. 29] suggests) as a continuation of other writings of Peirce on number theory, in particular in his efforts, in "On the Logic of Number" [Peirce *1881a*] of 1881, which has been shown [Shields *1981*; *1997*] to be equivalent to the axiom systems for number theory of Julius Wilhelm Richard Dedekind (1831–1916) in [Dedekind *1888*] *Was sind und was sollen die Zahlen* and Giuseppe Peano (1858–1932) in [Peano *1889*] *Arithmetices principia, nova methodo exposita*;[4] in the manuscript "Fundamental Properties of Number" of 5 January 1886 [Peirce *1886*] in which the natural numbers are informally defined as the linear sequence of positive integers, built up by a successor operator and between which an inclusion relation either of more than and less than hold, according to which, if n has a property P, then $n + 1$ has property P if n is included in $n + 1$; and in the fragmentary manuscripts "Number" [Peirce *ca. 1887a*] and "Logic of Number" [Peirce *ca. 1887b*] of 1887, in which Peirce undertakes alternative axiomatizations of basic arithmetic, in his efforts to develop arithmetic within his algebraic logic.

There is a shift in Peirce's approach between these previous writings and the work in "Logical Studies of Number Theory", however. Although Peirce looks to algebraic logic formulating for the expressions of arithmetic equations, Peirce is now concerned to answer the proof-theoretic question, namely, attempting to determine whether the algebraic logic offers an algorithm for solving arithmetic equations, rather than merely a developing algebraic logic as a pasigraphy for writing out the equations and deriving them, one from the other. In short, Peirce now appears to be concerned now with using algebraic logic to develop an algorithm for numerical solutions for equations of higher arithmetic, rather than as an exercise in studying the logical structure of arithmetic equations. It may be that Peirce was stimulated in this new approach by the work he was doing for the *Century Dictionary*, and in particular for his work on the definition of "number" for the *Dictionary*.

There are two historically interesting and important anticipations which "Logical Studies of Number Theory" suggests, beyond that relating specifically to Hilbert's Ninth Problem, then: (1) the Tenth Problem, of whether there exists an algorithm for the solution in the positive integers of Diophantine equations; and (2) the concept of encoding the syntax of one theory in the language of another theory. The second procedure, of encoding, was developed by Gödel for his proof of the incompleteness of arithmetic. The gödelnumbers, as we know, of course, coded logical propositions of first-order logic number-theoretically, whereas Peirce's proposal for developing, or at least searching for, an algorithm for solving number-theoretic equations went in the direction of encoding number theoretic equations in the language of Boole's calculus—the only mathematical logic then available.

If our interpretation of what Peirce was hoping to accomplish in "Logical Studies of Number Theory", then he must be understood as anticipating an affirmative answer to what we now know as Hilbert's Tenth Problem. But we also know that in 1931 Gödel proved the incompleteness of arithmetic, and in particular of any axiomatic system capable of adequately formalizing number theory; and in 1970 Yurii Matiyasevich proved [Matiyasevich *1970*] that the answer to Hilbert's tenth problem is negative. It would be sheer speculation to suppose that Peirce failed to work through the details of the method he had begun to prepare to sketch because, at some point, he suspected, or even realized, that number theory is incomplete, or that there is no algorithm for solving all equations of number theory. What can, however, be said, is that Peirce inadvertently defined much of the work of mathematical logic of the twentieth century, and anticipated, if not the answers, then certainly the questions, that directed the work in mathematical logic of the twentieth century. But we still have to be careful here, even in our use of "inadvertently", since the pivotal manuscript remained unpublished, and, so far as the history of logic is concerned, it was not Hilbert, in his Congress talk and in his publications and foundational disputes, and not Peirce, who established the goals for the direction of mathematical logic in the twentieth century.

Looking closely at Hilbert's Tenth Problem, we see that it is concerned with the question of whether there exists an algorithm that will decide, in a finite number of steps, whether an arbitrary Diophantine equation has solutions, and, if so, how many roots (solutions) the equation has. In 1960, Martin David Davis (b. 1928), Hilary Whitehall Putnam (b. 1926), and Julia Bowman Robinson (1919–1985) proved [Davis, Putnam, & Robinson *1961*], in arguments combining logic and number theory, and the following year published, their conjecture on the basis of which it is to be concluded that no such algorithm can exist. Their colleague Yuri Vladimirovich Matiyasevich (b. 1947) definitively proved that the existence of such an algorithm is impossible, demonstrating that the Davis-Putnam-Robinson Conjecture is true [Matiyasevich *1970*]. (An account of the tenth problem and its solution was given by Matiyasevich [Matiyasevich *1993a*], as well as in English translation [Matiyasevich *1993b*]. Davis, Matiyasevich and Robinson also examined the significance of this result [Davis, Matiyasevich, & Robinson *1971*]. See also [Adams & Goldstein *1976*, 191*n*.] and [Grosswald *1984*, 257]. For nontechnical expositions, see [Davis *1973*].) He did so by recursively defining the Fibonacci sequence $a_0 = a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$. The short answer is that we know that Gödel's incompleteness theorem is correct, that formal axiomatic system adequate for number theory can prove ever true statement of number theory.

We are still left with the historiographical questions of whether Peirce really was suggesting in "Logical Studies of Number Theory" that, in translating number theory into the language of formal logic, it might be possible to find such an algorithm or set of algorithms; and whether his failure to pursue the matter further arose because of a recognition, or at least an intuition, that perhaps there in fact is no such algorithm. What I suggest is that, at the minimum, Peirce did have Hilbert's ninth and tenth problem. And, assuming that the dating of the manuscript is correct, he was at work on the problems almost a decade before they were posed by Hilbert.

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Notes

[1] A list of the definitions which Peirce contributed to the *Century Dictionary* is availabe at: http://www.pep.éuqam.ca/listesdesmots.pep. The Universite de Montréal à Québec's Projet d'édition Peirce (UQÀM) is selecting and preparing Peirce's contributions to the *Century Dictionary* as vol. 7 of the *Writings of Charles S. Peirce: A Chronological Edition*. Another list, prepared by the Institute for Studies in Pragmatism, "Peirce's *Century Dictionary* Definitions", available at http://www.pragmaticism.net/century.htm, is based on [Ketner *1977*].

[2] Quoted, [Dickson *1919-23*, II, 731], [Nagell *1951*, 252]; English translation on at [Ore *1988*, 204]: "It is impossible to write a cube as the sum of two cubes, a fourth power as the sum of two fourth powers and in general any power beyond the

second as the sum of two similar powers. For this I have discovered a truly wonderful proof, but the margin is too small to contain it."

[3] The axioms provided in [Peirce *1880/81*] are nearly identical to those which Peirce gave in his marginalia to [Peirce *1881*] and manuscripts, as described in [Ketner *1977*] No. 393 [Peirce *1881a*], untitled, and No. 394 "Continuous, simple quantity" [Peirce *1881c*]. For a detailed comparative description between [Peirce *1880/81*] and [Peirce *1881b*, *1881c*], see [Peirce *1989*], 575–576. These are the more salient manuscripts on number theory and the logic of number on which Peirce was working prior to turning to in "Logical Studies of Number Theory" [Peirce *ca*. *1890*] to the problem of finding an algorithm for solving diophantine equations and which (see [Houser *2010*, xliii, *n*. 28]).

[4] The similarities between Peirce's [*1881a*] and Dedekind's [*1888*] axiomatic systems sufficed to precipitate a charge of plagiarism by Peirce against Dedekind; see [Gana *1985*].

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